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A note on the impossibility of a satisfactory concept of stability for coalition formation games

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Abstract

We show that no solution to coalition formation games can satisfy a set of axioms that we propose as reasonable. Our result points out that "solutions" to the coalition formation cannot be interpreted as "resting points" in the way stable coalition structures are usually interpreted.

Keywords: Hedonic game; Coalition formation; Stability

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1. Introduction

A challenging question in game theory is not only to predict what coalitions will form in games where there is room for cooperation, but also reasons for the grand coalition not to be formed. A number of solutions have been proposed (see, for example, Chwe, 1994; Xue, 1998; Diamantoudi and Xue, 2003; Barberà and Gerber, 2003) but each one of them is subject to some criticism. Interestingly, this criticism does not aim at the plausibility of the coalition structures that are selected by the solution but rather at the underlying rationale for the selection. Most solutions for coalition formation games incorporate some notion of farsightedness while keeping the idea of stability that is inherent in the core. Coalitions are

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farsighted, if they take into account that any deviation can trigger further deviations so that the initial deviators may or may not end up in a situation which they prefer over the original one. Here lies the core of the problem faced by the proposed solution concepts: they appeal to the dynamic nature of coalition formation without specifying an order of moves or determining probabilities with which certain transitions are expected to arise.[1] Hence, it is easy to construct examples where the prediction of a solution seems unreasonable because it relies on the potential occurrence of a highly implausible and therefore unlikely chain of deviations.[2] A remedy clearly would be to model coalition formation as a noncooperative game, where the order of moves and the set of actions available to any coalition are clearly specified. However, this approach suffers from the fact that there is no naturally given game form for a coalition formation problem and that the set of Nash equilibria may be very sensitive to the specific details of the game.

Hence, in this note we go back to the tradition of cooperative game theory, which leaves out the details of the game and instead tries to characterize a solution by the properties of the coalition structures it selects. It turns out that already a short list of desirable requirements leads to an impossibility result: there does not exist any solution concept that satisfies all of them at the same time. The axioms we propose are compelling, whenever one views the elements of a solution as "resting points," in the sense that once they are reached, they will persist. Some authors present their solutions in a different vein: as part of a permanent rotation (Konishi and Ray, 2003) or as the set of coalition structures that will stick longer than others (Barberà and Gerber, 2003). Our result does not quarrel with these solutions, as long as they are properly interpreted, and not taken as predictions of coalition structures that will stay forever, once they are reached. Before engaging in much formalism, we consider, as an example, a simple hedonic game. This example will allow us to discuss informally all the issues involved, to introduce all the axioms we propose, and it will later be used in the formal proof of our result.

A *hedonic game* is given by $(N; (\succeq_i)_{i \in N})$, where N is the finite set of *players*, and $\succeq_i$ is a complete and transitive preference relation on $S_i(N) = \{S \subset N | i \in S\}$ for all $i \in N$. Strict preference will be denoted by $\succ_i$. A set $\phi \neq S \subset N$ is called a *coalition*. A *coalition structure* $\mathscr{S}$ on N is a partition of N into disjoint coalitions. By P we denote the set of all coalition structures on N. Let $S(i, \mathscr{S})$ be the coalition in $\mathscr{S} \in P$ that contains player i. Let $\mathbb{G}$ be a set of hedonic games. A *solution* on $\mathbb{G}$ is a correspondence Γ: $\mathbb{G} \twoheadrightarrow P$. Hence, a solution assigns to each hedonic game in $\mathbb{G}$ a (possibly empty) set of coalition structures $\mathscr{S} \in P$. Any element of $\Gamma(G)$ will be called *stable* for $G \in \mathbb{G}$.

Consider the following example of a three-player hedonic game, which is sometimes called the "roommate-problem."[3] Let players' preferences be such that

$$\begin{aligned}
&\{1,3\} \succ_1 \{1,2\} \succ_1 \{1\} \succ_1 \{1,2,3\}, \\
&\{1,2\} \succ_2 \{2,3\} \succ_2 \{2\} \succ_2 \{1,2,3\}, \qquad (1)\\
&\{2,3\} \succ_3 \{1,3\} \succ_3 \{3\} \succ_3 \{1,2,3\}.
\end{aligned}$$

[1] A notable exception is the approach taken by Konishi and Ray (2003) who study a dynamic process of coalition formation with endogenous transition probabilities.

[2] Chwe (1994) provides a number of examples illustrating this point for the largest consistent set.

[3] Observe that this game is different from Gale and Shapley's (1962) roommate problem, where four players can split into groups of two.

We will use a simplified notation for coalition structures and write, for example, [12|3] for the coalition structure $\{\{1,2\},\{3\}\}$. The possible coalition structures in this game are

$$[1|2|3],\ [12|3],\ [13|2],\ [23|1],\ [123].$$

Which coalition structures could be considered a solution in this game? First of all, for a coalition structure to belong to the solution it must be that this structure is not Pareto dominated by any other. The rationale is simple: if some agents are to gain from re-organizing the society while the rest are indifferent, then the status quo cannot be stable, since everyone would agree to that re-organization. In our example, Pareto efficiency rules out the grand coalition and the coalition structures where all agents are alone.

A second natural requirement on a solution is symmetry. Take a game, where we can permute the players such that the permuted game coincides with the original game. Then, if one coalition structure is stable, the permuted coalition structure should also be stable. By Pareto efficiency we had already discarded all coalition structures as candidates for stable structures except for those where two agents are together and the remaining one is alone. Symmetry then tells us that all these three coalition structures must be stable or none of them can be.

We have already mentioned that solutions to coalition formation games are problematic, and one of the reasons is because, if we want to have an all-embracing theory, it must always select at least one coalition structure as the solution. This requirement is also one of our axioms: for all games, we require that a solution concept provides for at least one stable coalition structure. In general, coalition structures in the core of the coalition formation game would be great candidates for a solution (though not necessarily the only ones). But since the core is empty unless one imposes strong assumptions on the domain of games (see Banerjee et al., 2001; Bogomolnaia and Jackson, 2002), it has rather limited predictive power. For our example, the only possibilities left by the preceding axioms were to either identify three stable coalition structures or none. Now we are left with only one option: that of retaining the three coalitional structures [12|3], [13|2], [23|1] as the solution for the game.

Yet, this candidate solution fails to meet our last requirement, which we will call *self-consistency* and which is reminiscent of the notion of internal stability in the definition of the Von Neumann–Morgenstern stable set (Von Neumann and Morgenstern, 1944). In a nutshell, what the condition requires is that there should be no obvious incentive for agents to break one coalition structure in the solution in favor of another coalition structure also belonging to the solution. Let us be more precise about the reasons for such a requirement. When comparing two coalition structures $\mathscr{S}$ and $\mathscr{S}'$ we may want to look at those agents, who would prefer to form a coalition $S' \in \mathscr{S}'$, rather than staying in those where they belong to in $\mathscr{S}$. These agents can be seen as those interested in moving from $\mathscr{S}$ to $\mathscr{S}'$. In general, though, there is no guarantee that these agents alone can enforce the passage from one coalition structure to the other. This is why we concentrate on those cases where the passage from $\mathscr{S}$ to $\mathscr{S}'$ is enforceable by the interested parties alone. There are two cases where enforceability is unquestionable. One is when all coalitions in $\mathscr{S}'$ are potential movers. The other case is when all coalitions in $\mathscr{S}'$ but one are potential movers, and the remaining coalition is a singleton: if all others move, then the leftover agent cannot do anything else but staying alone. Let us now state more formally our fourth requirement on a solution. If $\mathscr{S}$ and $\mathscr{S}'$ are stable, then it cannot be that a family of coalitions can enforce $\mathscr{S}'$ and that all their members prefer $\mathscr{S}'$ to $\mathscr{S}$. We are aware that this axiom may be more controversial than the rest. Hence, we would like to elaborate on its rationale. First of all, there may be many cases where the interests and the action of some agents alone cannot fully define the consequences for a change in coalition structures. This is why we limit attention to those cases where the interested agents can clearly enforce a change, as already mentioned before. Then, we claim that our requirement is necessary for a set of stable coalition structures to be interpretable as a set of resting points, because of the combination of two facts. One is that we demand the set of agents who improve away from a stable coalition structure to do so in one shot. The other is that we require the improvement to be attained at another coalition structure which is also stable. Notice that the existence of a one-shot gain is not enough to disqualify the starting coalition structure as being stable, since it could be that the ensuing change was transient, and that agents might be deterred by the threat of further change to the worse. But this would require the arrival structure after the first shot to be unstable, in contradiction with the assumption that it belongs to the solution.

If, on the contrary, we insist in considering the arrival structure as a final resting point, then we are left without arguments to consider the initial one as stable. Either one or the other coalition structure fails to pass the requirement. Notice that our requirement is weak, since it only involves limited (enforceable) moves and one-shot gains. A notion of stability meeting the requirement may still be criticized on other grounds: moves that are not enforceable in our restricted sense may still be considered possible, and more complex chains of disruptive moves may be conceived. Our point is that the proposed requirement is minimal, if one accepts our terms of reference.

If we now go back to our example, we can see that each one of our three candidates to be a stable coalition structure dominates one of the others. Hence, the choice of these three, which is required by the conjunction of the remaining axioms, is precluded by the last one: there cannot exist any solution satisfying all the axioms we have stated. In the following section we will formally present our impossibility result.

2. An impossibility theorem

Let $\mathscr{C}$ be a set of coalitions. Then $\mathscr{C}$ can *enforce* $\mathscr{S}\in P$, if

$$\mathscr{S} = \bigcup_{\mathscr{S}\in\mathscr{C}} \{S\} \quad \text{or } \mathscr{S}=\left(\bigcup_{S\in\mathscr{C}} \{S\}\right) \quad \cup\{i\} \text{ for some } i\in\text{N}.$$

We say that $\mathscr{S}'$ *dominates* $\mathscr{S}$, if there exists a set of coalitions $\mathscr{C}$, such that

1. $\mathscr{C}$ can enforce $\mathscr{S}'$,
2. For all $S\in\mathscr{C}$ either

$$S(i,\mathscr{S}')\succ_i S(i,\mathscr{S}) \quad \text{for all } i\in\text{S},$$

or

$$S\succ_i T \text{ for all } i\in S \text{ and for all } T\in S_i(N)\backslash\{S\},$$

i.e. either all members of $S\in\mathscr{C}$ strictly prefer $\mathscr{S}'$ over $\mathscr{S}$ or they strictly prefer S over any other coalition.[4]

[4] In the latter case S is a *top coalition* of N (see Banerjee et al., 2001) and the members of S have no reason to veto against the move from $\mathscr{S}$ to $\mathscr{S}'$.

Let Γ be a solution on some set of hedonic games $\mathbb{G}$. Consider the following axioms.

A1 (Nonemptiness). $\Gamma(G)\neq\emptyset$ for all $G\in\mathbb{G}$.

A2 (Symmetry). If $G=(N;\,(\succeq_i)_{i\in N})\in\mathbb{G}$ is such that there exists a permutation π on N with

$$S\succeq_i T\Leftrightarrow\pi(S)\succeq_{\pi(i)}\pi(T),$$

for all $i\in N$ and for all S, $T\in S_i(N)$, then

$$\mathcal{S}\in\Gamma(G)\Leftrightarrow\pi(\mathcal{S})\in\Gamma(G).$$ [5]

A3 (Pareto Optimality). Let $G\in\mathbb{G}$ and $\mathcal{S}\in\Gamma(G)$. Then there does not exist $\mathcal{S}'\in P$, such that $S(i,\mathcal{S}')\succeq_i S(i,\mathcal{S})$ for all $i\in N$, with a strict inequality for at least one i.

A4 (Self-Consistency). Let $G\in\mathbb{G}$ and $\mathcal{S}$, $\mathcal{S}'\in\Gamma(G)$. Then $\mathcal{S}$ does not dominate $\mathcal{S}'$ and $\mathcal{S}'$ does not dominate $\mathcal{S}$.

Let $\mathbb{D}$ be a set of hedonic games with player set N, that includes all games with strict preferences, i.e. all $(N;\,(\succeq_i)_{i\in N})$ such that for all $i\in N$ and all S, $T\in S_i(N)$ with $S\neq T$, we have $S\succ_i T$ or $T\succ_i S$.

Theorem 2.1. *If $\#N\geq 3$, then there does not exist a solution on $\mathbb{D}$ which satisfies* (A1)–(A4).

Proof. Consider the following game $G\in\mathbb{D}$ which is an embedding of our introductory example into an n-player game with $n\geq 3$. Let $N=\{1,2,\ldots,n\}$ and let preferences satisfy (1) and $\{i\}\succ_i S$ for all $i\in\{1,2,3\}$ and all $S\in S_i(N)\backslash\{i\}$ with $S\neq\{i,j\}$ for all $j\in\{1,2,3\}$, $j\neq i$. Moreover, let

$$\{i\}\succ_i S \quad \text{for all } \mathrm{S}\in\mathrm{S_i(N)}\setminus\{i\}, \text{ and for all } \mathrm{i}\notin\{1,2,3\}.$$

Then, by (A1), (A2) and (A3), $\Gamma(G)=\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3$, where $\mathcal{S}_1=[12|3|4|\ldots|n]$, $\mathcal{S}_2=[13|2|4|\ldots|n]$ and $\mathcal{S}_3=[23|1|4|\ldots|n]$. But then (A4) is violated since, for example, $\mathcal{S}_2$ dominates $\mathcal{S}_1$. □

3. Conclusion

Our paper emphasizes that it is impossible to define a fully satisfactory notion of stability for hedonic coalition formation games. This is the case if one interprets the fact that a coalition structure is stable as a prediction that the corresponding coalitions will persist, once they are formed. There are essentially two routes to take as a consequence of this impossibility result. One route is to abandon one of our axioms in order to recover the existence of a solution. However, it seems difficult to give preference to a subset of the axioms since all reflect very desirable properties. The other route is to attribute a different, probabilistic interpretation to the solution of a hedonic game, where the elements of the solution are not

[5] If π: $N\rightarrow N$ is a permutation, then we define $\pi(S):=\{j|j=\pi(i)$ for some $i\in S\}$ for any coalition S and $\pi(\mathcal{S}):=\{T|T=\pi(S)$ for some $S\in\mathcal{S}\}$ for any coalition structure $\mathcal{S}$.

considered to be persistent but rather to be the support of some long-run distribution on the set of coalition structures.[6] We hope that this note initiates a new discussion of solutions to coalition formation games along these lines.

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[6] This is in the spirit of Konishi and Ray (2003). Their model, however, requires that preferences can be expressed in a cardinal way, which is not the case for the class of hedonic games considered here.